

An analytic description of the damping of gravitational waves by free streaming neutrinos.

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We provide an analytic solution to the general wavelength integro-differential equation describing the damping of tensor modes of gravitational waves due to free streaming neutrinos in the early universe. Our result is expressed as a series of spherical Bessel functions whose coefficients are functions of the reduced wave number Q .

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1. INTRODUCTION

Observations of the cosmic microwave background (CMB) have given increasing support to inflationary cosmological models. Density perturbations during this inflationary period are believed to have given rise to the large scale structures of the universe [1]. In addition to these scalar perturbations, a spectrum of gravitational waves is also produced [2] (tensor modes) which could provide information about the early universe. In particular, the contribution of these tensor modes (measured in terms of a tensor-to-scalar ratio [3]) to the temperature anisotropy of the CMB could be used to check the predictions of inflationary models. Here we shall consider the effect of anisotropic inertia on the gravitational radiation and confirm that it is non-negligible. Although it will be assumed here that the anisotropic inertia is dominated by neutrinos and antineutrinos [4], Ref. [5] has proposed a free streaming relativistic gas contribution and a method to constrain its fraction density through measurements of the CMB B-polarization spectra.

In a 2004 paper [4], Weinberg derives an integro-differential equation for the propagation of cosmological gravitational waves. He writes a wave equation for the perturbation to the metric $h_{ij}(\mathbf{x}, t)$ and defines $\chi(u)$ as

$$h_{ij} = h_{ij}(0) \chi(u), \quad (1)$$

where u is the conformal time multiplied by the wave number

$$u = k \int^t \frac{dt'}{a(t')}. \quad (2)$$

The wave equation for the perturbation leads to an integro-differential equation which the function χ satisfies for general wavelengths. In the variable $y = a(t)/a_{EQ}$, where a_{EQ} is the expansion parameter at matter-radiation equality, this is Eq. (32) of [4]

$$(1+y)\chi''(y) + \left(\frac{2(1+y)}{y} + \frac{1}{2}\right)\chi'(y) + Q^2\chi(y) = -\frac{24f_\nu(0)}{y^2} \int_0^y K(y, y') \frac{d\chi(y')}{dy'} dy', \quad (3)$$

with the initial conditions:

$$\chi(0) = 1, \quad \chi'(0) = 0. \quad (4)$$

Here $f_\nu(0) = 0.40523$ is the fraction of the energy density in neutrinos and $Q = \sqrt{2}k/k_{EQ}$. The kernel K will be discussed below, and y is related to u in the following manner

$$u = 2Q \left(\sqrt{1+y} - 1 \right). \quad (5)$$

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We will provide an analytic solution of Eqs. (3) and (4) that is valid for general wavelengths. It will be shown that the effect of the anisotropic inertia is to damp the amplitude of the perturbation relative to the case where free-streaming neutrinos are absent, and that, in general, this damping depends on Q . Thus, it is important to have a solution capable of describing the damping for all wavelengths. For $Q^2 \gg 1$, it has been shown that the solution for $\chi(u)$ can be written as a series of spherical Bessel functions whose coefficients are independent of Q [6]. In [4], Weinberg analyzes the $Q^2 \gg 1$ limit, provides results for the damping factor when $Q^2 \ll 1$ and makes some observations about the damping for general value of Q . Our aim here is to show that it is possible to extend the spherical Bessel function expansion [6, 7] for $\chi(u)$ to the case of general Q . Having done this, we can recapture the $Q^2 \ll 1$ results of [4], the $Q^2 \gg 1$ results of [6] and indicate how to obtain results for intermediate values of Q .

In the next section, we derive the equation that must be satisfied by a spherical Bessel function expansion for $\chi(u)$, examine its form in the $Q^2 \gg 1$ and obtain a recurrence relation for general Q . In Section 3, we examine the long wavelength ($Q^2 \ll 1$) limit and determine the damping factor for this case. Section 4 contains a discussion of general wavelength case and we end with some conclusions. Lists of the various expansion coefficients and the details related to the solution of the recurrence relation are contained in the Appendix.

2. SOLUTION OF THE GENERAL WAVELENGTH EQUATION

In terms of the variable u , Eq. (3) becomes

$$\chi''(u) + \frac{4(u+2Q)}{u^2+4Qu}\chi'(u) + \chi(u) = -\frac{16CQ^2}{(u^2+4Qu)^2} \int_0^u K(u-u')\chi'(u')du', \quad (6)$$

with $C = 24f_\nu(0) = 9.72552$ and the initial conditions

$$\chi(0) = 1, \quad \chi'(0) = 0. \quad (7)$$

The kernel K is a linear combination of spherical Bessel functions

$$K(z) = \frac{1}{16} \int_{-1}^1 dx (1-x^2)^2 e^{ixz} = -\frac{\sin(z)}{z^3} - \frac{3\cos(z)}{z^4} + \frac{3\sin(z)}{z^5} = \frac{1}{15}j_0(z) + \frac{2}{21}j_2(z) + \frac{1}{35}j_4(z), \quad (8)$$

which suggests a solution to Eq. (6) exists in the form of a series of spherical Bessel functions. Indeed the work done in Ref. [6] on the short wavelength limit shows that the convolution of a series of spherical Bessel functions with the kernel returns another series of spherical Bessel functions, so we look for a solution of the form

$$\chi(u) = \sum_{n=0}^{\infty} \alpha_n j_n(u). \quad (9)$$

Multiplying Eq. (6) through by $(u^2+4Qu)^2$, expanding all the terms, and dividing by u^4 yields

$$\begin{aligned} \left(1 + \frac{8Q}{u} + \frac{16Q^2}{u^2}\right) \chi''(u) + \left(\frac{4}{u} + \frac{24Q}{u^2} + \frac{32Q^2}{u^3}\right) \chi'(u) \\ + \left(1 + \frac{8Q}{u} + \frac{16Q^2}{u^2}\right) \chi(u) = -\frac{16CQ^2}{u^4} \int_0^u K(u-u')\chi'(u')du'. \end{aligned} \quad (10)$$

The action of the differential operator on $j_n(u)$ gives:

$$\begin{aligned} \left[\left(1 + \frac{8Q}{u} + \frac{16Q^2}{u^2}\right) \frac{d^2}{du^2} + \left(\frac{4}{u} + \frac{24Q}{u^2} + \frac{32Q^2}{u^3}\right) \frac{d}{du} + \left(1 + \frac{8Q}{u} + \frac{16Q^2}{u^2}\right)\right] j_n(u) \\ = \frac{1}{u^4} [n(n+1)16Q^2 + n(n+2)8Qu + n(n+3)u^2] j_n(u) - \frac{1}{u^4} [2u^2(u+4Q)] j_{n+1}(u), \end{aligned} \quad (11)$$

and Eq. (10) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} \alpha_n [n(n+1)16Q^2 + n(n+2)8Qu + n(n+3)u^2] j_n(u) - 2 \sum_{n=0}^{\infty} \alpha_n (u^3 + 4Qu^2) j_{n+1}(u) \\ = -16CQ^2 \sum_{n=0}^{\infty} \alpha_n \int_0^u K(u-u') \frac{dj_n(u')}{du'} du'. \end{aligned} \quad (12)$$

If we divide through by the highest power, the Bessel function recurrence relation (AS 10.2.18)[8]

$$\frac{j_n(z)}{z} = \frac{j_{n-1}(z) + j_{n+1}(z)}{(2n+1)} \quad (13)$$

can be used to eliminate the powers of u . Dividing through by u^3 yields

$$\begin{aligned} \sum_{n=0}^{\infty} \alpha_n \left[\frac{16Q^2}{u^3} n(n+1) + \frac{8Q}{u^2} n(n+2) + \frac{1}{u} n(n+3) \right] j_n(u) - 2 \sum_{n=0}^{\infty} \alpha_n \left(1 + \frac{4Q}{u} \right) j_{n+1}(u) \\ = -\frac{16CQ^2}{u^3} \sum_{n=0}^{\infty} \alpha_n \int_0^u K(u-u') \frac{dj_n(u')}{du'} du'. \end{aligned} \quad (14)$$

2.1. Short Wavelength Limit

Eq. (13) can now be used iteratively to eliminate the powers of u and obtain an expansion purely in spherical Bessel functions. However before we proceed, let us check what happens in the short wavelength limit ($Q \gg 1$). In this limit the terms with Q^2 dominate and Eq. (14) becomes

$$\sum_{n=0}^{\infty} \alpha_n n(n+1) j_n(u) = -C \sum_{n=0}^{\infty} \alpha_n \int_0^u K(u-u') \frac{dj_n(u')}{du'} du'. \quad (15)$$

The derivative of $j_n(z)$ is given by (AS 10.2.19)[8]

$$\frac{dj_n(z)}{dz} = \frac{n j_{n-1}(z) - (n+1) j_{n+1}(z)}{(2n+1)}, \quad (16)$$

and Eq. (15) becomes

$$\sum_{n=0}^{\infty} \alpha_n n(n+1) j_n(u) = -C \sum_{n=0}^{\infty} \frac{\alpha_n}{(2n+1)} \int_0^u du' K(u-u') [n j_{n-1}(u') - (n+1) j_{n+1}(u')]. \quad (17)$$

We can shift indices on the Bessel functions if we define $\alpha_n = 0$ for $n < 0$ to obtain

$$\sum_{n=0}^{\infty} \alpha_n n(n+1) j_n(u) = -C \sum_{n=0}^{\infty} \omega_n \int_0^u du' K(u-u') j_n(u'), \quad (18)$$

with

$$\omega_n = \left[\frac{\alpha_{n+1}(n+1)}{(2n+3)} - \frac{n \alpha_{n-1}}{(2n-1)} \right]. \quad (19)$$

The convolution integral can be evaluated using the technique given in Ref. [6]

$$\sum_{\ell=0}^{\infty} \omega_{\ell} \int_0^u du' K(u-u') j_{\ell}(u') = \sum_{n=0}^{\infty} \epsilon_n j_n(u), \quad (20)$$

where

$$\epsilon_n = \frac{(2n+1)}{2} i^n \left(\sum_{\ell=0}^{\infty} \sum_{m=0,2,4} d_m \omega_{\ell} (-i)^{\ell+m+1} I_{n\ell}^m \right), \quad (21)$$

and

$$I_{n\ell}^m = \int_{-1}^1 ds P_n(s) [Q_m(s) P_{\ell}(s) + P_m(s) Q_{\ell}(s)], \quad (22)$$

with

$$d_0 = \frac{1}{15}, \quad d_2 = \frac{2}{21}, \quad d_4 = \frac{1}{35}. \quad (23)$$

Then, Eq. (18) becomes

$$\sum_{n=0}^{\infty} \alpha_n n(n+1) j_n(u) = -C \sum_{n=0}^{\infty} \epsilon_n j_n(u). \quad (24)$$

This yields the following recurrence relation

$$\alpha_n n(n+1) = -C \epsilon_n. \quad (25)$$

The ϵ_n can be found using Eqs. (21), (22) and (23). The first 15 are given in Appendix A. For $n = 0$, Eq. (25) vanishes, so α_0 is undetermined. However, the initial condition $\chi(0) = 1$ fixes it to unity. The next six equations are

$$\begin{aligned} 2\alpha_1 &= -\frac{C\alpha_1}{15} \\ 6\alpha_2 &= -C \left(\frac{\alpha_2}{15} - \frac{\alpha_0}{6} \right) \\ 12\alpha_3 &= -C \left(\frac{\alpha_3}{15} - \frac{\alpha_1}{18} \right) \\ 20\alpha_4 &= -C \left(-\frac{\alpha_0}{10} - \frac{\alpha_2}{20} + \frac{\alpha_4}{15} \right) \\ 30\alpha_5 &= -C \left(-\frac{11\alpha_1}{225} - \frac{33\alpha_3}{700} + \frac{\alpha_5}{15} \right) \\ 42\alpha_6 &= -C \left(-\frac{13\alpha_0}{900} - \frac{13\alpha_2}{315} - \frac{143\alpha_4}{3150} + \frac{\alpha_6}{15} \right) \end{aligned}$$

The first equation requires $\alpha_1 = 0$, and we see that all the odd terms depend recursively on α_1 . Thus, all the odd terms vanish. The constant C was defined as

$$C = 24f_{\nu}(0) = 9.72552. \quad (26)$$

Solving for the first 3 non-zero α_n ($n > 0$) gives

$$\begin{aligned} \alpha_2 &= 0.243807 \\ \alpha_4 &= 5.28424 \times 10^{-2} \\ \alpha_6 &= 6.13545 \times 10^{-3} \end{aligned}$$

in agreement with Ref. [6].

2.2. Recurrence Relation

Returning to the general case given by Eq. (14), we can apply Eq. (20) to the righthand side and obtain

$$\sum_{n=0}^{\infty} \alpha_n \left[\frac{16Q^2}{u^3} n(n+1) + \frac{8Q}{u^2} n(n+2) + \frac{1}{u} n(n+3) \right] j_n(u) - 2 \sum_{n=0}^{\infty} \alpha_n \left(1 + \frac{4Q}{u} \right) j_{n+1}(u) = -\frac{16CQ^2}{u^3} \sum_{n=0}^{\infty} \epsilon_n j_n(u). \quad (27)$$

Eq. (13) can then be used recursively to obtain (see Appendix C)

$$\sum_{n=0}^{\infty} [16Q^2 \beta_n + 8Q(\gamma_n - \theta_n) + \delta_n - 2\alpha_{n-1}] j_n(u) = -16CQ^2 \sum_{n=0}^{\infty} \lambda_n j_n(u). \quad (28)$$

where $\beta_n, \gamma_n, \delta_n, \theta_n, \lambda_n$ are defined in Appendix C and presented in Appendix D. Eq. (28) leads to the recurrence relation

$$16Q^2(\beta_n + C\lambda_n) + 8Q(\gamma_n - \theta_n) + \delta_n - 2\alpha_{n-1} = 0. \quad (29)$$

The equations given by Eq. (29) determine the coefficients of the spherical Bessel functions in Eq. (9). With these coefficients, Eq. (9) provides an analytic solution to the inhomogeneous Eq. (6) for general values of the wave number Q . Eq. (29) can also provide the coefficients of the homogeneous solution (the solution without free streaming neutrinos) by setting $C = 0$, yielding the following recurrence relation

$$16Q^2\beta_n + 8Q(\gamma_n - \theta_n) + \delta_n - 2\alpha_{n-1} = 0. \quad (30)$$

The coefficients were found by solving Eqs. (29) and (30) using *Mathematica* (with the condition $\alpha_n = 0$ for $n < 0$). The first 15 inhomogeneous and homogeneous coefficients are given in Appendix A. The a_n are the inhomogeneous coefficients and the b_n are the homogeneous coefficients. This convention will be used throughout the remainder of the paper.

3. THE LONG WAVELENGTH LIMIT

We found an analytic solution to Eq. (6) consisting of an expansion in spherical Bessel functions of the following form

$$\chi(u) = \sum_{n=0}^{\infty} \alpha_n(Q) j_n(u), \quad (31)$$

where the $\alpha_n(Q)$ are explicit functions of the wave number Q . The convention used here will be to replace $\alpha_n(Q)$ with $a_n(Q)$ for the inhomogeneous solution and $b_n(Q)$ for the homogeneous solution. Upon first glance, this expansion seems to be divergent in the limit $Q \rightarrow 0$ since the coefficients depend inversely on Q . However, the expansion is in fact finite due to the implicit presence of Q in $j_n(u)$. To make the low Q limit unambiguous, the Q dependence will be removed from the Bessel functions using the following theorem (AS 9.1.74)[8]

$$\lambda^{-\mu} j_{\mu}(\lambda z) = \sum_{m=0}^{\infty} \frac{(1 - \lambda^2)^m}{m!} \left(\frac{z}{2}\right)^m j_{\mu+m}(z), \quad (32)$$

where μ and λ can be arbitrary complex numbers. If we define $u = Qs$, where $s = 2(\sqrt{1+y} - 1)$, then we can write Eq. (31) as

$$\chi(s, Q) = \sum_{n=0}^{\infty} \alpha_n(Q) j_n(Qs). \quad (33)$$

Using Eq. (32) we can write

$$j_n(Qs) = Q^n \sum_{m=0}^{\infty} \frac{(1-Q^2)^m}{m!} \left(\frac{s}{2}\right)^m j_{n+m}(s), \quad (34)$$

so Eq. (33) becomes

$$\chi(s, Q) = \sum_{n=0}^{\infty} \alpha_n(Q) Q^n \sum_{m=0}^{\infty} \frac{(1-Q^2)^m}{m!} \left(\frac{s}{2}\right)^m j_{n+m}(s). \quad (35)$$

It is now convenient to define new coefficients $c_n(Q)$ (inhomogeneous) and $d_n(Q)$ (homogeneous) that are finite as $Q \rightarrow 0$

$$c_n(Q) = a_n(Q) Q^n \quad (36)$$

$$d_n(Q) = b_n(Q) Q^n \quad (37)$$

The first 15 c_n and d_n are given in Appendix B. The expansion in Eq. (35) becomes (in the inhomogeneous case)

$$\chi(s, Q) = \sum_{n=0}^{\infty} c_n(Q) \sum_{m=0}^{\infty} \frac{(1-Q^2)^m}{m!} \left(\frac{s}{2}\right)^m j_{n+m}(s) \quad (38)$$

For $Q \rightarrow 0$, only c_0 contributes and the canonical damping factor $|\chi(s_L)/\chi_0(s_L)|^2 \rightarrow 1$ since the homogeneous expansion is the same in this limit (i.e. only d_0 contributes and it is also 1). We would also like to examine the Weinberg damping factor $|\chi'(s_L)/\chi'_0(s_L)|^2$ and show that it is well defined and finite in the limit $Q \rightarrow 0$. Differentiating Eq. (38) with respect to s yields

$$\chi'(s, Q) = \sum_{n=0}^{\infty} c_n(Q) \sum_{m=0}^{\infty} \frac{(1-Q^2)^m}{2^m m!} \left\{ m s^{m-1} j_{n+m}(s) + s^m \left[\frac{(n+m)j_{n+m-1}(s) - (n+m+1)j_{n+m+1}(s)}{2(n+m)+1} \right] \right\}. \quad (39)$$

The cosmologically interesting value of s is when $y = 22.1 \Omega_M h^2$. This is the value of s at last scattering, denoted s_L . For $\Omega_M h^2 = 0.15$ we have $s_L = 2.15452$. Plugging this value of s into Eq. (39) and expanding using 50 terms in each sum yields

$$\chi'(s_L, Q) = -0.573659 Q^2 + 0.243294 Q^4 - 0.0381658 Q^6 + 0.00315913 Q^8 + \mathcal{O}(Q^{10}). \quad (40)$$

For $Q \ll 1$, the Q^2 term dominates and we have

$$\chi'(s_L, Q) \approx -0.573659 Q^2. \quad (41)$$

The $c_n(Q)$ in Eq. (39) can be replaced with the $d_n(Q)$ to obtain an expansion of the same type for the homogeneous solution. Using the same method, we obtain the following expansion

$$\chi'_0(s_L, Q) = -0.601268 Q^2 + 0.264497 Q^4 - 0.0424247 Q^6 + 0.00356526 Q^8 + \mathcal{O}(Q^{10}). \quad (42)$$

As before, for $Q \ll 1$, the Q^2 term dominates and we have

$$\chi'_0(s_L, Q) \approx -0.601268 Q^2 \quad (43)$$

so the Weinberg damping factor $|\chi'(s_L, Q)/\chi'_0(s_L, Q)|^2$ for $Q = 0$ can be computed as

$$\left| \frac{\chi'(s_L, 0)}{\chi'_0(s_L, 0)} \right|^2 = \lim_{Q \rightarrow 0} \left| \frac{-0.573659 Q^2}{-0.601268 Q^2} \right|^2 = 0.910273, \quad (44)$$

in agreement with Weinberg's discussion on the low Q limit. As Weinberg remarks "This damping is relatively insensitive to Q for small Q ." Confirmation of this statement is seen here where the damping is independent of Q until the fourth order terms become important. The finiteness of this damping factor shows that the solution is well defined for arbitrarily small values of Q .

The expansions given in Eqs. (38) and (39) are valid for general values of Q . However for $Q > 1$ the expansion quickly starts to fail with any finite number of terms, due to the fact that the large powers of Q overpower their coefficients and the final term in any finite sum dominates. Writing out Eqs. (38) and (39) using *Mathematica* (100 terms per sum) will give an expansion in powers of Q up to Q^{300} . As shown in Figure 1, this expansion is well behaved for $Q < 1$ and converges up to approximately $Q = 15$.

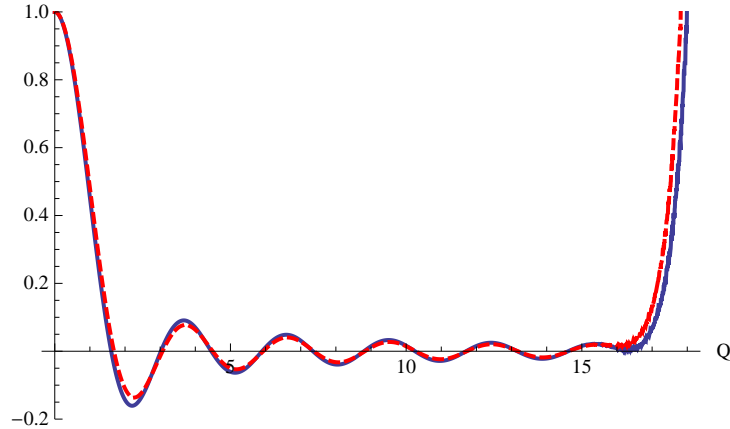


FIG. 1: $\chi_0(s_L, Q)$ (blue) and $\chi(s_L, Q)$ (red).

The ratios of χ 's and their derivatives can be used to plot the damping factors, as shown in Figure 2. These damping factors are the ratios of two oscillating functions and therefore become infinite at the zeros of the function in the denominator. However, since the functions are only slightly out of phase, these spikes occur where $\chi'(s_L)$ is small and according to Ref. [4], this makes the spikes uninteresting since the multipole coefficients for the corresponding values of ℓ will be very difficult to measure.

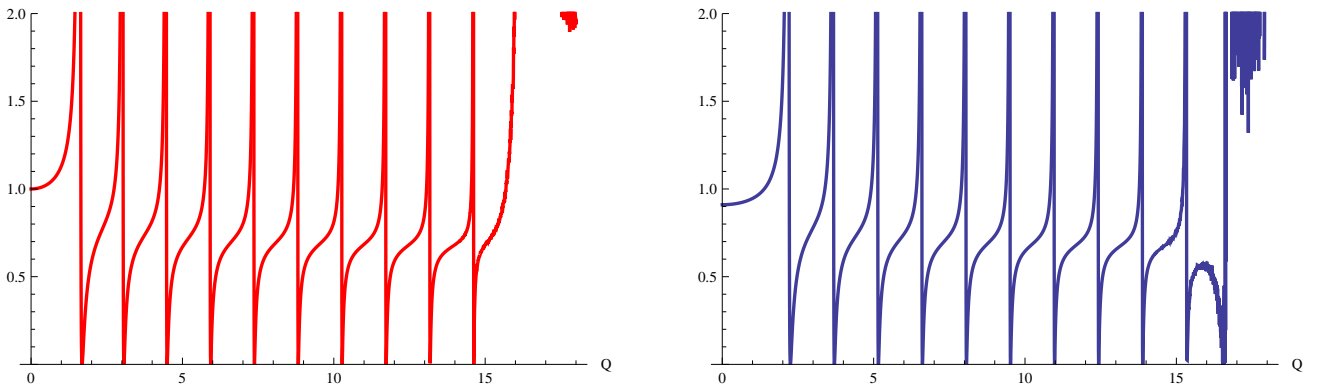


FIG. 2: The damping factor $|\chi(s_L)/\chi_0(s_L)|^2$ is shown in the left panel and $|\chi'(s_L)/\chi'_0(s_L)|^2$ is shown in the right panel.

Q	$ \chi(s_L)/\chi_0(s_L) ^2$	$ \chi'(s_L)/\chi'_0(s_L) ^2$
0	1	0.910273
0.01	1.00001	0.910275
0.1	1.00081	0.910561
0.55	1.02745	0.919509
0.8	1.06802	0.931291
1	1.13058	0.946036
5	0.669293	1.23524
10	0.811905	0.644909

TABLE I: Damping values for various Q 's from Eq. (38) are listed.

4. DAMPING FOR GENERAL WAVELENGTHS

Though the expansion given in Eq. (38) is valid for general values of Q , it is cumbersome to use in practice. However, now that the solution has been shown to be finite in the long wavelength limit, we can return to a more useful expansion of the form given in Eq. (33),

$$\chi(s, Q) = \sum_{n=0}^{\infty} \alpha_n(Q) j_n(Qs). \quad (45)$$

Here $s = 2(\sqrt{1+y} - 1)$ and the $\alpha_n(Q)$ should be replaced with the $a_n(Q)$ ($b_n(Q)$) for the inhomogeneous (homogeneous) solution. As long as numerical applications of Eq. (45) explicitly include the Q in the argument of the Bessel functions, this expansion reproduces the damping found in the previous section (given in Table I) as $Q \rightarrow 0$, despite the fact that the coefficients diverge. This is due to the fact that $j_n(Qs)$ goes like Q^n for small Q , removing the divergences produced by the coefficients. For $Q > 1$, there is no difficulty and asymptotically this expansion reproduces the damping found in Ref. [6] for $Q \gg 1$, as shown in Figure 3.

Unless otherwise specified, 100 terms were used in all numerical evaluations of Eq. (45) occurring in this paper. The observation in Ref. [4] that the reduction in $|\chi'(s_L)/\chi'_0(s_L)|^2$ from 1 for $Q = 0.55$ and $Q = 0.8$ is about 8% and 7% respectively is confirmed in Table II. Also shown in this Table is the trend of $|\chi'(s_L)/\chi'_0(s_L)|^2$ in the relatively flat regions between the spikes steadily decrease from the value ≈ 0.9 for $Q \ll 1$ to a value close to 0.644 for $Q \approx 10$.

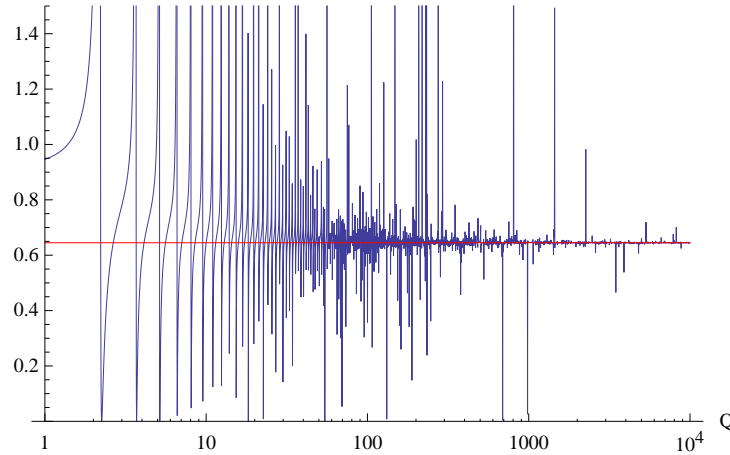


FIG. 3: The damping factor $|\chi'(s_L)/\chi'_0(s_L)|^2$ is shown along with the horizontal line representing $A^2 = 0.645019$.

Q	$ \chi(s_L)/\chi_0(s_L) ^2$	$ \chi'(s_L)/\chi'_0(s_L) ^2$
0.01	1.00001	0.910275
0.1	1.00081	0.910561
0.55	1.02745	0.919509
0.8	1.06802	0.931291
1	1.13058	0.946036
10	0.811905	0.644909
10^2	0.64902	0.866294
10^3	0.646315	0.642628
10^4	0.644227	0.64506
10^5	0.645004	0.645055
10^6	0.645013	0.645008

TABLE II: Damping values for various Q 's from Eq. (45)

5. CONCLUSIONS

We have shown that the treatment of gravitational wave damping by free streaming neutrinos can be framed in terms of a series of spherical Bessel functions for all values of the reduced wave number Q . The result for the coefficients of the spherical Bessel series when $Q \gg 1$ [6] emerges quite simply from the coefficient recurrence relation for a general Q . For the opposite limit, $Q \ll 1$, an identity is used to remove the Q^n -dependence from $j_n(Qs)$ and the analysis can then be extended to arbitrarily small values of Q , including the non-leading behavior of the series. Once Q is large enough, the coefficients obtained from the general recurrence relation can be used directly, even though they contain inverse powers of Q .

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Appendix A: Expansion Coefficients

The relations between the $\epsilon_n(Q)$ and the expansion coefficients $a_n(Q)$ are listed below for $n = 0$ to 14.

$$\begin{aligned}
\epsilon_0(Q) &= 0 \\
\epsilon_1(Q) &= \frac{a_1(Q)}{15} \\
\epsilon_2(Q) &= \frac{a_2(Q)}{15} - \frac{a_0(Q)}{6} \\
\epsilon_3(Q) &= \frac{a_3(Q)}{15} - \frac{a_1(Q)}{18} \\
\epsilon_4(Q) &= \frac{a_4(Q)}{15} - \frac{a_2(Q)}{20} - \frac{a_0(Q)}{10} \\
\epsilon_5(Q) &= \frac{a_5(Q)}{15} - \frac{33a_3(Q)}{700} - \frac{11a_1(Q)}{225} \\
\epsilon_6(Q) &= \frac{a_6(Q)}{15} - \frac{143a_4(Q)}{3150} - \frac{13a_2(Q)}{315} - \frac{13a_0(Q)}{900} \\
\epsilon_7(Q) &= \frac{a_7(Q)}{15} - \frac{13a_5(Q)}{294} - \frac{11a_3(Q)}{294} - \frac{5a_1(Q)}{588} \\
\epsilon_8(Q) &= \frac{a_8(Q)}{15} - \frac{17a_6(Q)}{392} - \frac{221a_4(Q)}{6300} - \frac{17a_2(Q)}{2520} + \frac{17a_0(Q)}{11025} \\
\epsilon_9(Q) &= \frac{a_9(Q)}{15} - \frac{323a_7(Q)}{7560} - \frac{19a_5(Q)}{567} - \frac{4693a_3(Q)}{793800} + \frac{19a_1(Q)}{18900} \\
\epsilon_{10}(Q) &= \frac{a_{10}(Q)}{15} - \frac{19a_8(Q)}{450} - \frac{17a_6(Q)}{525} - \frac{41a_4(Q)}{7560} + \frac{a_2(Q)}{1260} - \frac{a_0(Q)}{2520} \\
\epsilon_{11}(Q) &= \frac{a_{11}(Q)}{15} - \frac{23a_9(Q)}{550} - \frac{437a_7(Q)}{13860} - \frac{69989a_5(Q)}{13721400} + \frac{299a_3(Q)}{436590} - \frac{23a_1(Q)}{83160} \\
\epsilon_{12}(Q) &= \frac{a_{12}(Q)}{15} - \frac{115a_{10}(Q)}{2772} - \frac{5a_8(Q)}{162} - \frac{18715a_6(Q)}{3841992} + \frac{85a_4(Q)}{137214} - \frac{a_2(Q)}{4536} + \frac{25a_0(Q)}{174636} \\
\epsilon_{13}(Q) &= \frac{a_{13}(Q)}{15} - \frac{15a_{11}(Q)}{364} - \frac{69a_9(Q)}{2275} - \frac{887a_7(Q)}{188760} + \frac{13319a_5(Q)}{23123100} - \frac{1751a_3(Q)}{9249240} + \frac{a_1(Q)}{9555} \\
\epsilon_{14}(Q) &= \frac{a_{14}(Q)}{15} - \frac{261a_{12}(Q)}{6370} - \frac{145a_{10}(Q)}{4851} - \frac{2850091a_8(Q)}{624323700} + \frac{8207a_6(Q)}{15057900} - \frac{551a_4(Q)}{3243240} \\
&\quad + \frac{29a_2(Q)}{343035} - \frac{29a_0(Q)}{463320}
\end{aligned}$$

Using $C = 24f_\nu(0) = 9.72552$, the first fifteen $a_n(Q)$ are

$$\begin{aligned}
a_0(Q) &= 1 \\
a_1(Q) &= 0 \\
a_2(Q) &= 0.243807 \\
a_3(Q) &= \frac{0.843856}{Q} \\
a_4(Q) &= 0.0528424 - \frac{1.31658}{Q^2} \\
a_5(Q) &= \frac{2.60385}{Q^3} - \frac{1.98354}{Q} \\
a_6(Q) &= 0.00613545 - \frac{6.16442}{Q^4} + \frac{8.15545}{Q^2} \\
a_7(Q) &= \frac{16.7669}{Q^5} - \frac{31.6633}{Q^3} + \frac{2.81654}{Q} \\
a_8(Q) &= 0.000297534 - \frac{50.564}{Q^6} + \frac{130.451}{Q^4} - \frac{22.4418}{Q^2} \\
a_9(Q) &= \frac{162.551}{Q^7} - \frac{583.689}{Q^5} + \frac{141.989}{Q^3} - \frac{3.66375}{Q} \\
a_{10}(Q) &= 0.0000616273 - \frac{524.266}{Q^8} + \frac{2845.09}{Q^6} - \frac{868.128}{Q^4} + \frac{46.8401}{Q^2} \\
a_{11}(Q) &= \frac{1458.97}{Q^9} - \frac{15066.7}{Q^7} + \frac{5429.37}{Q^5} - \frac{433.843}{Q^3} + \frac{4.49219}{Q} \\
a_{12}(Q) &= -4.998662 \times 10^{-6} - \frac{1012.45}{Q^{10}} + \frac{86302.4}{Q^8} - \frac{35476.}{Q^6} + \frac{3653.68}{Q^4} - \frac{83.7481}{Q^2} \\
a_{13}(Q) &= -\frac{39324.1}{Q^{11}} - \frac{532143.}{Q^9} + \frac{244196.}{Q^7} - \frac{30143.3}{Q^5} + \frac{1064.03}{Q^3} - \frac{5.3138}{Q} \\
a_{14}(Q) &= 2.33661 \times 10^{-6} + \frac{570180.}{Q^{12}} + \frac{3.5156 \times 10^6}{Q^{10}} - \frac{1.7758 \times 10^6}{Q^8} + \frac{251558.}{Q^6} \\
&\quad - \frac{11768.6}{Q^4} + \frac{135.604}{Q^2}
\end{aligned}$$

For large Q , the coefficients of the $a_{2n}(Q)$ reduce to the results in Ref. [6]. For a fixed value of y , these coefficients and the $b_n(Q)$ below yield finite results. This is illustrated in Figure 4.

The coefficients for the homogeneous solution are

$$\begin{aligned}
b_0(Q) &= 1 \\
b_1(Q) &= 0 \\
b_2(Q) &= 0 \\
b_3(Q) &= \frac{1.45833}{Q} \\
b_4(Q) &= -\frac{2.95313}{Q^2} \\
b_5(Q) &= \frac{7.57969}{Q^3} - \frac{2.56667}{Q} \\
b_6(Q) &= -\frac{23.4609}{Q^4} + \frac{12.4583}{Q^2} \\
b_7(Q) &= \frac{84.8364}{Q^5} - \frac{55.7289}{Q^3} + \frac{3.61607}{Q} \\
b_8(Q) &= -\frac{350.539}{Q^6} + \frac{261.588}{Q^4} - \frac{31.5725}{Q^2} \\
b_9(Q) &= \frac{1628.06}{Q^7} - \frac{1323.89}{Q^5} + \frac{218.214}{Q^3} - \frac{4.64444}{Q} \\
b_{10}(Q) &= -\frac{8391.91}{Q^8} + \frac{7261.36}{Q^6} - \frac{1451.12}{Q^4} + \frac{63.3263}{Q^2} \\
b_{11}(Q) &= \frac{47522.4}{Q^9} - \frac{43105.3}{Q^7} + \frac{9831.73}{Q^5} - \frac{624.673}{Q^3} + \frac{5.66288}{Q} \\
b_{12}(Q) &= -\frac{293207.}{Q^{10}} + \frac{275999.}{Q^8} - \frac{69352.2}{Q^6} + \frac{5591.11}{Q^4} - \frac{110.739}{Q^2} \\
b_{13}(Q) &= \frac{1.9574 \times 10^6}{Q^{11}} - \frac{1.89854 \times 10^6}{Q^9} + \frac{513799.}{Q^7} - \frac{48915.7}{Q^5} + \frac{1475.69}{Q^3} - \frac{6.67582}{Q} \\
b_{14}(Q) &= -\frac{1.4056 \times 10^7}{Q^{12}} + \frac{1.39746 \times 10^7}{Q^{10}} - \frac{4.01079 \times 10^6}{Q^8} + \frac{431978.}{Q^6} - \frac{17104.2}{Q^4} + \frac{176.825}{Q^2}
\end{aligned}$$

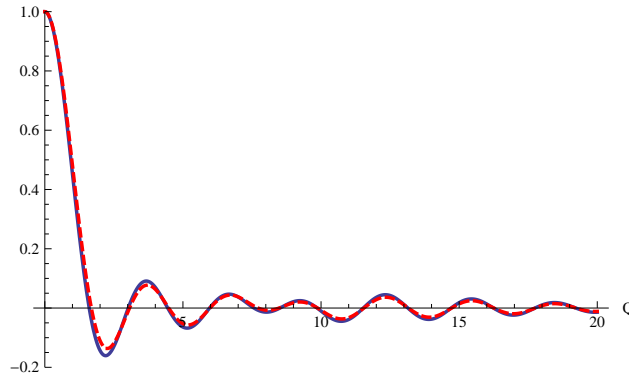


FIG. 4: The plots for $\chi(s_L, Q)$ (dashed) and $\chi_0(s_L, Q)$ (solid) are shown using the 15 coefficients listed above.

Appendix B: Long Wavelength Finite Expansion Coefficients

The coefficients for the inhomogeneous solution are

$$\begin{aligned}
c_0(Q) &= 1 \\
c_1(Q) &= 0 \\
c_2(Q) &= 0.243807Q^2 \\
c_3(Q) &= 0.843856Q^2 \\
c_4(Q) &= Q^2 (0.0528424Q^2 - 1.31658) \\
c_5(Q) &= Q^2 (2.60385 - 1.98354Q^2) \\
c_6(Q) &= Q^2 (0.00613545Q^4 + 8.15545Q^2 - 6.16442) \\
c_7(Q) &= Q^2 (2.81654Q^4 - 31.6633Q^2 + 16.7669) \\
c_8(Q) &= Q^2 (0.000297534Q^6 - 22.4418Q^4 + 130.451Q^2 - 50.564) \\
c_9(Q) &= Q^2 (-3.66375Q^6 + 141.989Q^4 - 583.689Q^2 + 162.551) \\
c_{10}(Q) &= Q^2 (0.0000616273Q^8 + 46.8401Q^6 - 868.128Q^4 + 2845.09Q^2 - 524.266) \\
c_{11}(Q) &= Q^2 (4.49219Q^8 - 433.843Q^6 + 5429.37Q^4 - 15066.7Q^2 + 1458.97) \\
c_{12}(Q) &= Q^2 (-4.99866 \times 10^{-6}Q^{10} - 83.7481Q^8 + 3653.68Q^6 - 35476Q^4 + 86302.4Q^2 - 1012.45) \\
c_{13}(Q) &= Q^2 (-5.3138Q^{10} + 1064.03Q^8 - 30143.3Q^6 + 244196Q^4 - 532143Q^2 - 39324.1) \\
c_{14}(Q) &= Q^2 (2.33661 \times 10^{-6}Q^{12} + 135.604Q^{10} - 11768.6Q^8 + 251558Q^6 - 1.7758 \times 10^6Q^4 + 3.5156 \times 10^6Q^2 + 570180)
\end{aligned}$$

The coefficients for the homogeneous solution are

$$\begin{aligned}
d_0(Q) &= 1 \\
d_1(Q) &= 0 \\
d_2(Q) &= 0 \\
d_3(Q) &= 1.45833Q^2 \\
d_4(Q) &= -2.95313Q^2 \\
d_5(Q) &= Q^2 (7.57969 - 2.56667Q^2) \\
d_6(Q) &= Q^2 (12.4583Q^2 - 23.4609) \\
d_7(Q) &= Q^2 (3.61607Q^4 - 55.7289Q^2 + 84.8364) \\
d_8(Q) &= Q^2 (-31.5725Q^4 + 261.588Q^2 - 350.539) \\
d_9(Q) &= Q^2 (-4.64444Q^6 + 218.214Q^4 - 1323.89Q^2 + 1628.06) \\
d_{10}(Q) &= Q^2 (63.3263Q^6 - 1451.12Q^4 + 7261.36Q^2 - 8391.91) \\
d_{11}(Q) &= Q^2 (5.66288Q^8 - 624.673Q^6 + 9831.73Q^4 - 43105.3Q^2 + 47522.4) \\
d_{12}(Q) &= Q^2 (-110.739Q^8 + 5591.11Q^6 - 69352.2Q^4 + 275999Q^2 - 293207) \\
d_{13}(Q) &= Q^2 (-6.67582Q^{10} + 1475.69Q^8 - 48915.7Q^6 + 513799Q^4 - 1.89854 \times 10^6Q^2 + 1.9574 \times 10^6) \\
d_{14}(Q) &= Q^2 (176.825Q^{10} - 17104.2Q^8 + 431978Q^6 - 4.01079 \times 10^6Q^4 + 1.39746 \times 10^7Q^2 - 1.4056 \times 10^7)
\end{aligned}$$

In contrast to Figure 1, using this set of 15 coefficients in Eq. (38) gives a convergent expansion to about $Q = 5$, as illustrated in Figure 5. This is adequate to treat the $Q \ll 1$ region.

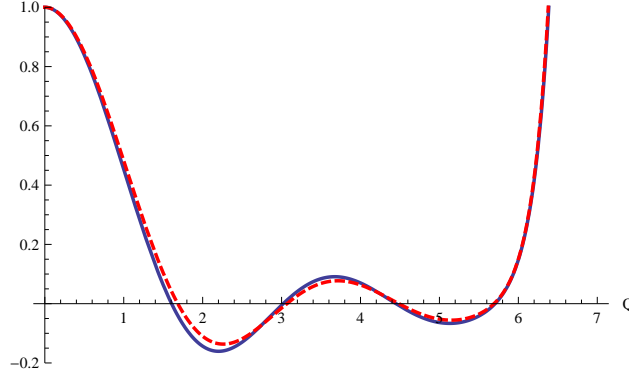


FIG. 5: The plots for $\chi(s_L, Q)$ (dashed) and $\chi_0(s_L, Q)$ (solid) are shown using the 15 coefficients listed above in Eq. (38).

Appendix C: Recurrence Calculation

We want to eliminate powers of u from

$$\sum_{n=0}^{\infty} \alpha_n \left[\frac{16Q^2}{u^3} n(n+1) + \frac{8Q}{u^2} n(n+2) + \frac{1}{u} n(n+3) \right] j_n(u) - 2 \sum_{n=0}^{\infty} \alpha_n \left(1 + \frac{4Q}{u} \right) j_{n+1}(u) = -\frac{16CQ^2}{u^3} \sum_{n=0}^{\infty} \epsilon_n j_n(u) \quad (C1)$$

The Bessel function recurrence relation

$$\frac{j_n(z)}{z} = \frac{j_{n-1}(z) + j_{n+1}(z)}{(2n+1)} \quad (C2)$$

can be used recursively to obtain the following relations:

$$\frac{j_n(z)}{z^2} = \frac{j_{n-2}(z)}{(2n-1)(2n+1)} + \frac{2j_n(z)}{(2n-1)(2n+3)} + \frac{j_{n+2}(z)}{(2n+1)(2n+3)} \quad (C3)$$

$$\begin{aligned} \frac{j_n(z)}{z^3} &= \frac{j_{n-3}(z)}{(2n+1)(2n-1)(2n-3)} + \frac{3j_{n-1}(z)}{(2n-3)(2n+1)(2n+3)} \\ &+ \frac{3j_{n+1}(z)}{(2n-1)(2n+1)(2n+5)} + \frac{j_{n+3}(z)}{(2n+1)(2n+3)(2n+5)} \end{aligned} \quad (C4)$$

Consider the first term on the LHS of Eq. (C1)

$$\begin{aligned} \sum_{n=0}^{\infty} \alpha_n \left[\frac{16Q^2}{u^3} n(n+1) + \frac{8Q}{u^2} n(n+2) + \frac{1}{u} n(n+3) \right] j_n(u) \\ = 16Q^2 \sum_{n=0}^{\infty} \alpha_n n(n+1) \frac{j_n(u)}{u^3} + 8Q \sum_{n=0}^{\infty} \alpha_n n(n+2) \frac{j_n(u)}{u^2} + \sum_{n=0}^{\infty} \alpha_n n(n+3) \frac{j_n(u)}{u} \end{aligned} \quad (C5)$$

The first term in Eq. (C5) can be re-written as follows with the aid of Eq. (C4):

$$\begin{aligned} 16Q^2 \sum_{n=0}^{\infty} \alpha_n n(n+1) \frac{j_n(u)}{u^3} &= 16Q^2 \sum_{n=0}^{\infty} \alpha_n n(n+1) \left[\frac{j_{n-3}(z)}{(2n+1)(2n-1)(2n-3)} + \frac{3j_{n-1}(z)}{(2n-3)(2n+1)(2n+3)} \right. \\ &\quad \left. + \frac{3j_{n+1}(z)}{(2n-1)(2n+1)(2n+5)} + \frac{j_{n+3}(z)}{(2n+1)(2n+3)(2n+5)} \right] \end{aligned} \quad (C6)$$

If we define $\alpha_n = 0$ for $n < 0$ then the indices on the Bessel functions can be shifted to obtain:

$$16Q^2 \sum_{n=0}^{\infty} \alpha_n n(n+1) \frac{j_n(u)}{u^3} = 16Q^2 \sum_{n=0}^{\infty} j_n(u) \left[\frac{\alpha_{n+3} (n+3)(n+4)}{(2n+3)(2n+5)(2n+7)} + \frac{3\alpha_{n+1} (n+1)(n+2)}{(2n-1)(2n+3)(2n+5)} \right. \\ \left. + \frac{3\alpha_{n-1} n(n-1)}{(2n-3)(2n-1)(2n+3)} + \frac{\alpha_{n-3} (n-3)(n-2)}{(2n-5)(2n-3)(2n-1)} \right] \quad (C7)$$

So we can write:

$$16Q^2 \sum_{n=0}^{\infty} \alpha_n n(n+1) \frac{j_n(u)}{u^3} = 16Q^2 \sum_{n=0}^{\infty} \beta_n j_n(u) \quad (C8)$$

with:

$$\beta_n = \left[\frac{\alpha_{n+3} (n+3)(n+4)}{(2n+3)(2n+5)(2n+7)} + \frac{3\alpha_{n+1} (n+1)(n+2)}{(2n-1)(2n+3)(2n+5)} \right. \\ \left. + \frac{3\alpha_{n-1} n(n-1)}{(2n-3)(2n-1)(2n+3)} + \frac{\alpha_{n-3} (n-3)(n-2)}{(2n-5)(2n-3)(2n-1)} \right] \quad (C9)$$

Following the same procedure for the rest of Eq. (C5) using Eqs. (C2) and (C3) yields:

$$\sum_{n=0}^{\infty} \alpha_n \left[\frac{16Q^2}{u^3} n(n+1) + \frac{8Q}{u^2} n(n+2) + \frac{1}{u} n(n+3) \right] j_n(u) = \sum_{n=0}^{\infty} j_n(u) [16Q^2 \beta_n + 8Q \gamma_n + \delta_n] \quad (C10)$$

with:

$$\gamma_n = \left[\frac{\alpha_{n+2} (n+2)(n+4)}{(2n+3)(2n+5)} + \frac{2\alpha_n n(n+2)}{(2n-1)(2n+3)} + \frac{\alpha_{n-2} n(n-2)}{(2n-3)(2n-1)} \right] \quad (C11)$$

$$\delta_n = \left[\frac{\alpha_{n+1} (n+1)(n+4)}{(2n+3)} + \frac{\alpha_{n-1} (n-1)(n+2)}{(2n-1)} \right] \quad (C12)$$

Similarly, the second term on the LHS of Eq. (C1) becomes:

$$-2 \sum_{n=0}^{\infty} \alpha_n \left(1 + \frac{4Q}{u} \right) j_{n+1}(u) = -2 \sum_{n=0}^{\infty} j_n(u) [4Q \theta_n + \alpha_{n-1}] \quad (C13)$$

with:

$$\theta_n = \left[\frac{\alpha_n}{(2n+3)} + \frac{\alpha_{n-2}}{(2n-1)} \right] \quad (C14)$$

So Eq. (C1) becomes:

$$\sum_{n=0}^{\infty} [16Q^2 \beta_n + 8Q(\gamma_n - \theta_n) + \delta_n - 2\alpha_{n-1}] j_n(u) = -\frac{16CQ^2}{u^3} \sum_{n=0}^{\infty} \epsilon_n j_n(u) \quad (C15)$$

With the condition $\alpha_n = 0$ for $n < 0$. The ϵ_n are already zero for $n < 0$, so the same method as above can be applied to the RHS of Eq. (C15). Using Eq. (C4) gives:

$$-\frac{16CQ^2}{u^3} \sum_{n=0}^{\infty} \epsilon_n j_n(u) = -16CQ^2 \sum_{n=0}^{\infty} j_n(u) \left[\frac{\epsilon_{n+3}}{(2n+3)(2n+5)(2n+7)} + \frac{3\epsilon_{n+1}}{(2n-1)(2n+3)(2n+5)} \right. \\ \left. + \frac{3\epsilon_{n-1}}{(2n-3)(2n-1)(2n+3)} + \frac{\epsilon_{n-3}}{(2n-5)(2n-3)(2n-1)} \right] \quad (C16)$$

So the RHS of Eq. (C15) becomes:

$$-\frac{16CQ^2}{u^3} \sum_{n=0}^{\infty} \epsilon_n j_n(u) = -16CQ^2 \sum_{n=0}^{\infty} \lambda_n j_n(u) \quad (\text{C17})$$

with:

$$\lambda_n = \left[\frac{\epsilon_{n+3}}{(2n+3)(2n+5)(2n+7)} + \frac{3\epsilon_{n+1}}{(2n-1)(2n+3)(2n+5)} \right. \\ \left. + \frac{3\epsilon_{n-1}}{(2n-3)(2n-1)(2n+3)} + \frac{\epsilon_{n-3}}{(2n-5)(2n-3)(2n-1)} \right] \quad (\text{C18})$$

Substituting, Eq. (C15) becomes

$$\sum_{n=0}^{\infty} [16Q^2\beta_n + 8Q(\gamma_n - \theta_n) + \delta_n - 2\alpha_{n-1}] j_n(u) = -16CQ^2 \sum_{n=0}^{\infty} \lambda_n j_n(u) \quad (\text{C19})$$

which is Eq. (28) of this paper.

Appendix D: Recurrence Coefficients

$$\beta_n = \left[\frac{\alpha_{n+3}(n+3)(n+4)}{(2n+3)(2n+5)(2n+7)} + \frac{3\alpha_{n+1}(n+1)(n+2)}{(2n-1)(2n+3)(2n+5)} \right. \\ \left. + \frac{3\alpha_{n-1}n(n-1)}{(2n-3)(2n-1)(2n+3)} + \frac{\alpha_{n-3}(n-3)(n-2)}{(2n-5)(2n-3)(2n-1)} \right] \quad (\text{D1})$$

$$\gamma_n = \left[\frac{\alpha_{n+2}(n+2)(n+4)}{(2n+3)(2n+5)} + \frac{2\alpha_n n(n+2)}{(2n-1)(2n+3)} + \frac{\alpha_{n-2}n(n-2)}{(2n-3)(2n-1)} \right] \quad (\text{D2})$$

$$\delta_n = \left[\frac{\alpha_{n+1}(n+1)(n+4)}{(2n+3)} + \frac{\alpha_{n-1}(n-1)(n+2)}{(2n-1)} \right] \quad (\text{D3})$$

$$\theta_n = \left[\frac{\alpha_n}{(2n+3)} + \frac{\alpha_{n-2}}{(2n-1)} \right] \quad (\text{D4})$$

$$\lambda_n = \left[\frac{\epsilon_{n+3}}{(2n+3)(2n+5)(2n+7)} + \frac{3\epsilon_{n+1}}{(2n-1)(2n+3)(2n+5)} \right. \\ \left. + \frac{3\epsilon_{n-1}}{(2n-3)(2n-1)(2n+3)} + \frac{\epsilon_{n-3}}{(2n-5)(2n-3)(2n-1)} \right] \quad (\text{D5})$$